Chap 2 Sums

(recurrence)

- $S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k = \sum_{0 \le k \le n} a_k \iff \begin{cases} S_0 = a_0 \\ S_n = S_{n-1} + a_n, \ (n \ge 1) \end{cases}$

(1) $\sum_{k \in K} c a_k = c \sum_{k \in K} a_k$;

(2) $\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$;

(3) $\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$.

p: 1-1, $j \xrightarrow{p} 2j+1 = k$

j	k
0	1
1	3
2	5

$\sum_{k=1,3,5} k^2 \stackrel{k \leftarrow 2j+1}{=\!=} \sum_{\substack{2j+1=1,3,5 \\ j=0,1,2}} (2j+1)^2$

(4) $\sum_{j \in J} \sum_{k \in K} a_{j,k} = \sum_{\substack{j \in J \\ k \in K}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$

(5) $\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$

- Change of index $k \leftarrow p(k)$ $\begin{cases} k \leftarrow k \pm 1 & \text{(peturbation)} \\ k \leftarrow n+1-k & \text{(reverse ordering)} \end{cases}$

- Example 1 $S_n = \sum_{1 \le k \le n} k$ $\quad (= 1 + 2 + \cdots + n)$

$\stackrel{k \leftarrow n+1-k}{=\!=} \sum_{\substack{1 \le n+1-k \le n \\ 1 \le k \le n}} (n+1-k)$ $\quad (= n + (n-1) + \cdots + 1)$

$2 S_n = \sum_{1 \le k \le n} (n+1) = n(n+1)$, $\quad S_n = \frac{1}{2} n(n+1) \quad = \binom{n+1}{2}$

- Example 2 $R_n = 1 + r + r^2 + \cdots + r^{n-1} = \sum_{0 \le k \le n-1} r^k \stackrel{k \leftarrow k-1}{=\!=} \sum_{\substack{0 \le k-1 \le n-1 \\ 1 \le k \le n}} r^{k-1}$

$r R_n = \sum_{1 \le k \le n} r^k = \sum_{0 \le k \le n-1} r^k + r^n - 1 = R_n + r^n - 1$, $\quad R_n = \frac{1 - r^n}{1 - r}$

- $\square_n = 1^2 + 2^2 + \cdots + n^2 = \sum_{1 \le k \le n} k^2 = ?$

(i) $k \leftarrow n+1-k$, $\square_n = \sum_{1 \le n+1-k \le n} (n+1-k)^2 = \sum_{1 \le k \le n} \left[(n+1)^2 - 2(n+1)k + k^2 \right] = n(n+1)^2 - 2(n+1) S_n + \square_n$

(ii) $k \leftarrow k+1$, $\square_n = \sum_{1 \le k+1 \le n} (k+1)^2 = \sum_{0 \le k \le n-1} (k^2 + 2k + 1) = (\square_n - n^2) + 2 S_{n-1} + n$

$\Rightarrow S_n = \frac{1}{2} n(n+1)$

• Example 3 $\square_n = 1^2 + 2^2 + \cdots + n^2 = \sum_{1 \le k \le n} k^2$

(0) (Lookup table)

(1) (Guess and Prove by induction)

(2) (change index) $\boxed{\square}_n = 1^3 + 2^3 + \cdots + n^3 = \sum_{1 \le k \le n} k^3 \underset{\overline{}}{\overset{k \leftarrow k+1}{=\!=}} \sum_{\substack{1 \le k+1 \le n \\ 0 \le k \le n-1}} (k^3 + 3k^2 + 3k + 1)$

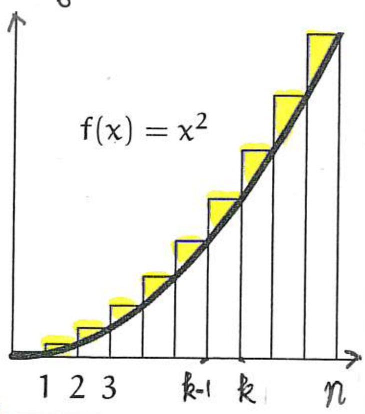
$$= (\boxed{\square}_n - n^3) + 3(\square_n - n^2) + 3 S_{n-1} + n \Rightarrow \underline{\square_n = \tfrac{1}{6} n(n+1)(2n+1)}$$

(3) (repertoire) $(\alpha = \beta = \gamma = 0,\ \delta = 1,\ \square_n = D(n))$

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n + \delta n^2 \ (*) \end{cases} \Rightarrow R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

$(\alpha, \beta, \gamma, \delta)$	$R_n = $ Eq	$(*)$
$(1,0,0,0)$	$1 = A(n)$	$1 = 1 \quad + \beta + \gamma n + \delta n^2$
$(0,1,0,0)$	$n = B(n)$	$n = n-1 \quad + \beta + \gamma n + \delta n^2$
$(0,-1,2,0)$	$n^2 = -B(n) + 2C(n)$	$n^2 = n^2 - 2n + 1 \quad + \beta + \gamma n + \delta n^2$
$(0,1,-3,3)$	$n^3 = B(n) - 3C(n) + 3D(n)$	$n^3 = n^3 - 3n^2 + 3n - 1 \quad + \beta + \gamma n + \delta n^2$

$$\Rightarrow A(n) = 1,\quad B(n) = n,\quad C(n) = \tfrac{1}{2}(n^2 + n),\quad D(n) = \frac{n(n+1)(2n+1)}{6}$$



(4) (Integration)

$$\square_n = \int_0^n x^2\,dx + \sum_{1 \le k \le n} \left(k^2 - \int_{k-1}^{k} x^2\,dx\right)$$

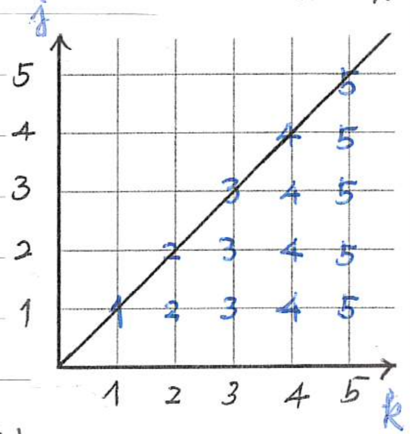
$$= \tfrac{1}{3} n^3 + \sum_{1 \le k \le n} \left[k^2 - \tfrac{1}{3}\left(k^3 - (k-1)^3\right)\right] = \frac{n^3}{3} - \sum_{1 \le k \le n} \left(k - \tfrac{1}{3}\right)$$

(5) (Expand + Contract) $(1 \le j \le k \le n)$

$$\square_n = \sum_{1 \le k \le n} k^2 = \sum_{1 \le k \le n} \sum_{1 \le j \le k} k = \sum_{1 \le j \le n} \sum_{j \le k \le n} k$$

$$= \sum_{1 \le j \le n} \frac{j+n}{2}(n-j+1) = \frac{1}{2} \sum_{1 \le j \le n} (n^2 - j^2 + n + j)$$

$$2\square_n = n(n^2+n) - \square_n + S_n$$



(6) (Finite calculus)

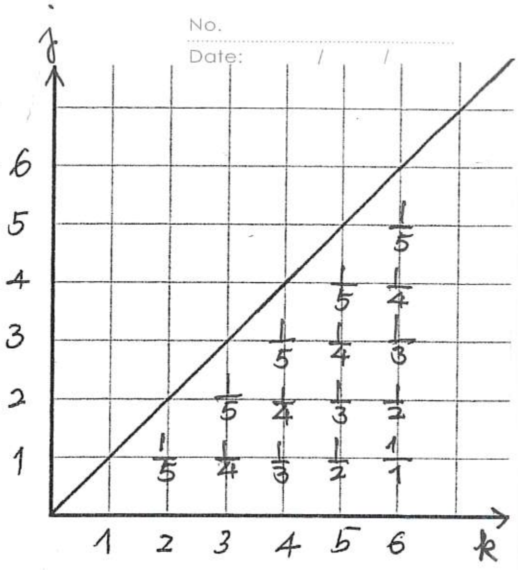
(7) (Binomial Coeff) $\sum_{1 \le k \le n} k^2 = \sum_{1 \le k \le n} 2\binom{k}{2} + \binom{k}{1} = 2\binom{n+1}{3} + \binom{n+1}{2}$

(8) (Generating functions) $f(x) = \sum_{0 \le k \le n} x^k,\quad (x f'(x))' = \sum_{1 \le k \le n} k^2 x^{k-1}$

(9) (Euler summation formula) $\quad = \frac{1 - x^{n+1}}{1 - x}$

(10) (Bernoulli numbers)

$(n=6)$



- **Example 4** 求 $\displaystyle\sum_{1\le j<k\le n}\frac{1}{n+j-k}$

原式 $\displaystyle= \sum_{1<k\le n}\ \sum_{1\le j<k}\frac{1}{n+j-k}$ $\quad (j\leftarrow k-j)$

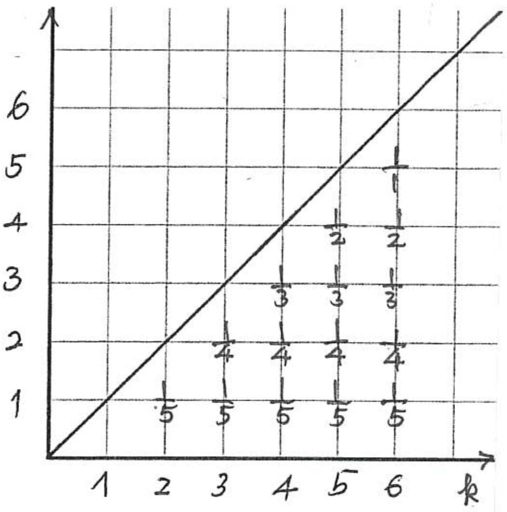
$\displaystyle= \sum_{1<k\le n}\ \sum_{1\le k-j<k}\frac{1}{n-j}$

$\displaystyle= \sum_{1<k\le n}\ \sum_{1\le j<k}\frac{1}{n-j}$

$\displaystyle= \sum_{1\le j<n}\ \sum_{j<k\le n}\frac{1}{n-j} = \sum_{1\le j<n}\frac{n-j}{n-j}$

$= \underline{\underline{(n-1)}}$

$\Downarrow\ j\leftarrow k-j$



- **Harmonic number**: $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$

- **Example 5** 求 $\displaystyle\sum_{1\le j<k\le n}\frac{1}{k-j} = \sum_{1\le k<n} H_k = \sum_{1\le j<n} H_j$

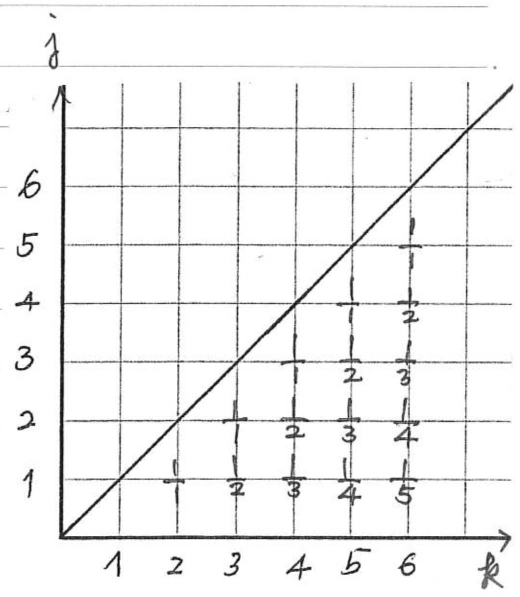
原式 $\displaystyle= \sum_{1<k\le n}\ \sum_{1\le j<k}\frac{1}{k-j}$ $\quad (j\leftarrow k-j)$

$\displaystyle= \sum_{1<k\le n}\ \sum_{1\le k-j<k}\frac{1}{j}$

$\displaystyle= \sum_{1<k\le n}\ \sum_{1\le j<k}\frac{1}{j}$

$\displaystyle= \sum_{1\le j<n}\ \sum_{j<k\le n}\frac{1}{j} = \sum_{1\le j<n}\frac{n-j}{j}$

$= \underline{\underline{nH_n - n}}$ $\quad (6.67)$



- $$\begin{aligned}
& 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\
H_1 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\
H_2 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\
H_3 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\
H_4 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\
H_5 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}
\end{aligned}$$

$H_1+H_2+\cdots+H_5 = 6H_6-6$

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- Summation factor: $s_n = \dfrac{a_{n-1}\cdots a_1(a_0)}{b_n b_{n-1}\cdots b_1(b_0)}$

(1) $T_n = b_n T_{n-1} + C_n \Rightarrow \underbrace{\dfrac{T_n}{b_n\cdots b_1(b_0)}}_{S_n} = \underbrace{\dfrac{T_{n-1}}{b_{n-1}\cdots b_1(b_0)}}_{S_{n-1}} + d_n \Rightarrow S_n = S_0 + \sum_{1\le k\le n} d_k$

(2) $a_n T_n = b_n T_{n-1} + C_n$

$$T_n = \frac{b_n}{a_n}T_{n-1} + \frac{C_n}{a_n} \Rightarrow \frac{1}{\frac{b_n}{a_n}\frac{b_{n-1}}{a_{n-1}}\cdots\frac{b_1}{a_1}}T_n = \frac{1}{\frac{b_{n-1}}{a_{n-1}}\cdots\frac{b_1}{a_1}}T_{n-1} + d_n$$

$$\Rightarrow \underbrace{\frac{a_n a_{n-1}\cdots a_1}{b_n b_{n-1}\cdots b_1}T_n}_{S_n} = \underbrace{\frac{a_{n-1}\cdots a_1}{b_{n-1}\cdots b_1}T_{n-1}}_{S_{n-1}} + d_n$$

- Example 1 $\begin{cases} T_0 = 0 \\ T_n = 2T_{n-1}+1 \quad (n\ge 1)\end{cases}$ $\quad s_n = \dfrac{1}{2^n}$

$$\frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$$
$$S_n = S_{n-1} + \frac{1}{2^n} \quad \left(d_n = \frac{1}{2^n}\right)$$
$$T_n = 2^n S_n = 2^n\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)$$
$$= 2^n - 1$$

- Example 2 (Quicksort) $\begin{cases} C_0 = 0 \\ C_n = (n+1) + \frac{2}{n}\sum_{0\le k\le n-1} C_k \quad (n\ge1)\end{cases}$

$$nC_n = n(n+1) + 2\sum_{0\le k\le n-1} C_k$$
$$-)\ (n-1)C_{n-1} = n(n-1) + 2\sum_{0\le k\le n-2} C_k$$

$\begin{cases} a_n = n \\ b_n = n+1 \end{cases}$

$$nC_n - (n-1)C_{n-1} = 2n + 2C_{n-1}$$
$$nC_n = (n+1)C_{n-1} + 2n \qquad s_n = \frac{(n-1)\cdots 1}{(n+1)\,n\cdots 2} = \frac{1}{(n+1)n}$$
$$\frac{C_n}{n+1} = \frac{C_{n-1}}{n} + \frac{2}{n+1} \qquad \left(d_n = \frac{2}{n+1}\right)$$

$$\frac{C_n}{n+1} = \frac{C_0}{1} + \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{n+1} = 2\left(H_n + \frac{1}{n+1} - 1\right)$$

$$C_n = 2(n+1)H_n - 2n$$

- Example 3 $\quad \begin{cases} a_m = m \\ b_m = n-m+1 \end{cases}$

$$mS_m = (n-m+1)S_{m-1} - \binom{n}{m} \quad (m\ge1), \qquad s_m = \frac{(m-1)\cdots 1}{(n-m+1)(n-m+2)\cdots n} = \frac{1}{m\binom{n}{m}}$$

$$\frac{S_m}{\binom{n}{m}} = \frac{S_{m-1}}{\binom{n}{m-1}} - \frac{1}{m}$$
$$= S_0 - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right)$$
$$= S_0 - H_m \Rightarrow S_m = \binom{n}{m}(S_0 - H_m)$$

Calculus

・定義

$$\begin{cases} Df(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} \\ \displaystyle\int_a^b g(x)\,dx = \lim_{\min \Delta x_i \to 0} \sum_{0\le i<n} g(x_i)(x_{i+1}-x_i) \qquad (b>a) \end{cases}$$

・定理1 (Fundamental Theorem)

$$g(x) = Df(x) \iff \int_a^b g(x)\,dx = f(x)\Big|_a^b = f(b)-f(a)$$

・定理2

- $D(\alpha f(x) + \beta g(x)) = \alpha Df(x) + \beta Dg(x)$
- $\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$
- $\int_a^a f(x)\,dx = 0$

  $\int_a^b + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

・定理3

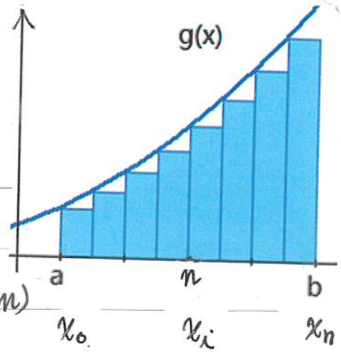
- $D(x^k) = kx^{k-1}$
- $D(\ln x) = \frac{1}{x}$
- $D(e^x) = e^x$
- $\int_a^b x^k\,dx = \begin{cases} \frac{x^{k+1}}{k+1}\Big|_a^b & k \ne -1 \\ \ln x \Big|_a^b & k=-1 \end{cases}$

・定理4 (Integration-by-parts)

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Discrete Calculus

(Difference)



・定義

$$\begin{cases} \Delta f(n) = f(n+1) - f(n) \\ \displaystyle\sum\nolimits_a^b g(n)\Delta n = \sum_{a\le n<b} g(n) \end{cases} \qquad (\delta n)$$

・定理1 (Fundamental Theorem)

$$g(n) = \Delta f(n) \iff \sum\nolimits_a^b g(n)\Delta n = f(n)\Big|_a^b$$

・定理2

- $\Delta(\alpha f(n) + \beta g(n)) = \alpha \Delta f(n) + \beta \Delta g(n)$
- $\sum_a^b \alpha f(n) + \beta g(n)\,\Delta n = \alpha \sum_a^b f(n)\Delta n + \beta \sum_a^b g(n)\Delta n$
- $\sum_a^a f(n)\Delta n = 0$

  $\sum_a^b + \sum_b^c f(n)\Delta n = \sum_a^c f(n)\Delta n$

・定理3

$$\begin{cases} n^{\underline{k}} = \overbrace{n(n-1)\cdots(n-k+1)}^{k\text{項}} \\ n^{\underline{-k}} = \dfrac{1}{(n+1)(n+2)\cdots(n+k)} \quad (k>0) \end{cases}$$

- $\Delta(n^{\underline{k}}) = kn^{\underline{k-1}}$
- $\Delta(H_n) = n^{\underline{-1}} = \frac{1}{n+1}$
- $\Delta(2^n) = 2^n$
- $\sum_a^b n^{\underline{k}}\Delta n = \begin{cases} \frac{n^{\underline{k+1}}}{k+1}\Big|_a^b & k\ne -1 \\ H_n \Big|_a^b & k=-1 \end{cases}$

・定理4 (Summation-by-parts)

$E(V_k) = V_{k+1}$

$$\sum\nolimits_a^b u\Delta V = uV\Big|_a^b - \sum\nolimits_a^b E(V)\Delta u$$

證

$d(uv) = u\,dv + v\,du$

(1) $\sum_{0\le k<n} (V_{k+1}-V_k)u_k = u_nV_n - u_0V_0 - \sum_{0\le k<n} V_{k+1}(u_{k+1}-u_k)$

(2) $\Delta(uV) = u_{k+1}V_{k+1} - u_kV_k - u_kV_{k+1} + u_kV_{k+1} = u\Delta V + E(V)\Delta u$

(3) $(n=3)$ $(V_1-V_0)u_0 + (V_2-V_1)u_1 + (V_3-V_2)u_2 = \cancel{u_3V_3} - u_0V_0 - V_1(u_1-u_0) - V_2(u_2-u_1) - V_3(u_3-u_2)$

證明:

• **定理1** $g(n) = \Delta f(n) = f(n+1) - f(n) \Leftrightarrow \sum_a^b g(n)\Delta n = \sum_{a\le n<b} g(n) = f(n)\Big|_a^b = f(b) - f(a)$

$f(n)$	$f(1)\ f(2)\ f(3)\ f(4)\ f(5)\ f(6)$	里程數(海拔高度), 存款餘額
$g(n)$	$g(2)\ g(3)\ g(4)\ g(5)$	距離 (樓層高), 收入

$$g(2)+g(3)+g(4)+g(5) = f(6)-f(2)$$

• **定理3**

$f(n)$	$n^{\underline{-3}}$	$n^{\underline{-2}}$	$n^{\underline{-1}}$	H_n	1	$n^{\underline{1}}$	$n^{\underline{2}}$	$n^{\underline{3}}$	$n^{\underline{k}}$
$\Delta f(n)$	$-3n^{\underline{-4}}$	$-2n^{\underline{-3}}$	$-n^{\underline{-2}}$	$n^{\underline{-1}}$	0	1	$2n^{\underline{1}}$	$3n^{\underline{2}}$	$kn^{\underline{k-1}}$

$n^{\underline{3}} = n(n-1)(n-2)$ $\qquad \Delta n^{\underline{3}} = (n+1)^{\underline{3}} - n^{\underline{3}} = (n+1)n(n-1) - n(n-1)(n-2) = 3n(n-1) = 3n^{\underline{2}}$

$n^{\underline{2}} = n(n-1)$

$n^{\underline{1}} = n$

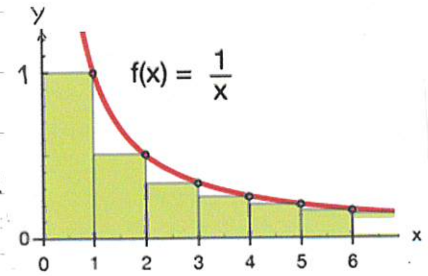
$n^{\underline{0}} = 1$

$n^{\underline{-1}} = \frac{1}{n+1}$

$n^{\underline{-2}} = \frac{1}{(n+1)(n+2)}$

$n^{\underline{-3}} = \frac{1}{(n+1)(n+2)(n+3)}$ $\qquad \Delta n^{\underline{-3}} = (n+1)^{\underline{-3}} - n^{\underline{-3}} = \frac{1}{(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)} = \frac{-3}{(n+1)(n+2)\cdots(n+4)} = -3n^{\underline{-4}}$

$$\begin{cases} \ln n = \int_1^n \frac{1}{x}dx \\ H_n = \sum_{1\le k\le n} \frac{1}{k} = \sum_0^n k^{\underline{-1}}\Delta k \end{cases}$$



$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $\qquad \Delta H_n = H_{n+1} - H_n = \frac{1}{n+1}$

• **Example 1** $\sum_{1\le k<n} k = \sum_1^{n+1} k^{\underline{1}}\Delta k = \frac{k^{\underline{2}}}{2}\Big|_1^{n+1} = \frac{(n+1)n}{2} - \frac{1\cdot 0}{2} = \underline{\underline{\frac{n(n+1)}{2}}}$

$\sum_{1\le k<n} k^2 = \sum_1^{n+1} k^{\underline{2}} + k^{\underline{1}}\Delta k = \frac{k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2}\Big|_1^{n+1} = \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \underline{\underline{\frac{n(n+1)(2n+1)}{6}}}$

• **Example 2** (p.56) $\sum_{0\le k\le n} k2^k = \sum_0^{n+1} \underbrace{k}_{u}\underbrace{2^k\Delta k}_{\Delta v} = k2^k\Big|_0^{n+1} - \sum_0^{n+1} 2^{k+1}\Delta k = (n+1)2^{n+1} - 2\cdot 2^k\Big|_0^{n+1}$

$\qquad = (n+1)2^{n+1} - 2\cdot 2^{n+1} + 2$

$\qquad = \underline{\underline{(n-1)2^{n+1} + 2}}$

$$\begin{cases} u = k & \Delta u = 1\cdot\Delta k \\ \Delta v = 2^k\Delta k & v = 2^k, \quad E(v) = 2^{k+1} \end{cases}$$

• **Example 3** (p.41) $\sum_{1\le k<n} H_k = \sum_1^n \underbrace{H_k}_{u}\underbrace{\Delta k}_{\Delta v} = kH_k\Big|_1^n - \sum_1^n (k+1)\frac{1}{k+1}\Delta k = (nH_n - 1) - (n-1)$

$\qquad = \underline{\underline{nH_n - n}} \qquad (6.67)$

$$\begin{cases} u = H_k & \Delta u = \frac{1}{k+1}\Delta k \\ \Delta v = \Delta k, & v = k, \quad E(v) = k+1 \end{cases}$$